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Common Fixed Point Theorems for Finite Number of Mappings without Continuity and Compatibility on Fuzzy 2-Metric Space

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Abstract

We prove related fixed point theorem for finite number of discontinuous, noncompatible mapping on noncomplete fuzzy 2-metric space.

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Introduction

Many authors have introduced the concept of fuzzy metric space in different way including [1], [2], [3] and [4]. Sessa [5], jungck [6] was introduced the concept of weak commutativity as an improvement of commutativity and Mishra et.al [7]. Compatible maps to type (β) by cho, pathak, Kang and Jung[9]. Sehgal, V.M. [10] investigated 2-metric space. Sharma [11], [12] and Sharma and Tiwari[13] proved common fixed point theorem in fuzzy metric space, fuzzy 2-metric space and fuzzy, 3-metric space without taking any function continuous.

In this paper, we prove a related fixed point theorem for finite number of discontinuous, noncompatible mapping on noncomplete fuzzy 2-metric space.

Preliminaries

Definition 2.1

A binary operation $*$:[0,1]× [0,1]→ [0,1] is called a continuous t-norm. If ([0, 1], $*$) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ are in [0, 1].

Definition 2.2

[13] The 3-tuple $(X, M,*)$ is called a fuzzy 2-metric space if X is arbitraty set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 * [0,\infty)$ satisfying the following conditions for all $x, y, z, u \; \epsilon$ X and $t_1, t_2, t_3 > 0$.

1. $M(x, y, z, 0) = 0$
2. $M(x, y, z, t) = 1\,, t > 0$ and when at least two of the tree points are equal.
3. $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$(symmetry about tree variables)
4. $M(x, y, z,\; t_1 + t_2 + t_3\,) \geq \; M(x, y, u, t_1) + M(x, u, z, t_2\,) * M(u, y, z, t_3)$ (this corresponds to tetrahedron inequality in 2-metric space)
   The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.
5. $M(x, y, z)$: $[0,1] \;\rightarrow\; [0,1]$ is left continuous.

Remark

Since $*$ is continuous it follows from (condition-4) that the limit of the sequence in FM-space is uniquely determined.

Let $(X, M,*)$ is a fuzzy 2-metric space with the following condition.

6. $\lim_{n\to\infty} M\,(x, y, a, t) = 1$ for all $x, y, a \;\in\; X$

Definition 2.3

[13] let $(X, M,*)$ is a fuzzy 2-metric space.

1. A sequence $\{x_{n\}}$ in fuzzy 2-metric space X is said to be convergent to a point x X, if $\lim_{n\to\infty} M\,(x_n\,, x, a, t) = 1 \quad a \in\; X\, and\; t > 0$
2. A sequence $\{x_{n\}}$ in fuzzy 2-metric space X is called a Cauchy sequence if $\lim_{n\to\infty} M\left(x_{n+p}\,, x_n, a, t\right) = 1 \quad a \in\; X\, and\; t > 0, p > 0$
3. A fuzzy 2-metric space in which every Cauchy sequence is convergent said to be complete.

Definition 2.4

A pair of mappings A and S is called weakly compatible in fuzzy 2-metric space if they commute at coincidence points.

Lemma 1

[13] For all $x, y, a \in X, M(x, y, a)$ is non-decreasing.

Lemma 2

[13] let { $y_{n\}}$ be a sequence in a fuzzy metric space $(X, M,*)$ with the condition (FM-6). If there exists a number $k \in (0,1)$ such that

$$M(y_{n+2}, y_{n+1}, a, kt) \geq M(y_{n+1}, y_n, a, t)$$

For all t>0 and n=1,2 , ……….. then $\{y_{n\}}$ is a Cauchy sequence in X.

Lemma 3

[13] if A for all $x, y, a \in X, t > 0$ and for a number $k \in (0,1)$,

$$M(x, y, a, kt) \geq M(x, y, a, t) \; then \; x = y.$$

In the following example, we know that every metric induces a fuzzy metric.

Example 1

[14], [15] let (X, d) be a metric space. Define $a * b = ab$ or $a * b = min\{a, b\}$ and for all $x, y, a \in X, t > 0$,

$$M(x, y, a, t) = (t/t + d(x, y, a)))$$

Then $(X, M,*)$ is a fuzzy 2-metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition(2.5)

[13] let $(X, M,*)$ be a fuzzy metric space and let A, B be self mappings of X. The mappings A and B are said to be compatible if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, a\,, t\,) = 1,$$

For all $a \in X$ and $t > 0$, whenever $\{x_{n\}}$ is a sequence in X such that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, a, t) = 1,$$

For all a X and t>0, whenever is a sequence in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Remark

1. In [16], [17] and [18] we can find the equivalent formulations of definitions of compatible maps, compatible maps of type (α) and compatible maps of type (β). Such maps are independent of each other and more general than commuting and weakly commuting maps ([19], [20]).
2. Compatible or compatible of type (α) or compatible of type (β) maps are weakly compatible but converse need not true (21).

Main Results

Theorem

Let (X,M,*) be an fuzzy 2-metric space. Let A,B,S,T,I,J,L,U,P and Q be mappings from X into itself such that

(1.1) $P(X) \subset ABIL(X), Q(X) \subset STJU(X),$

(1.2) There exists a constant k (0,1) such that

$$[1 + pM(STJUx, ABILy, a, kt)] * M^{2s}\,(Px, Qy, a, kt)] \geq p[M(Px, STJUx, a, kt) * M(Qy, ABILy, a, kt) + M(Qy, STJUx, a, kt) * M(Px, ABILy, a, kt)] + \min\{M^{2s}\,(ABILy, STJUx, a, t),\; M^q\,(Px, STJUx, a, t).\; M^{q'}(Qy, ABILy, a, t), M^r\,(Qy, SIJUx, a, \alpha t).\; M^{r'}(Px, ABILy, a, (2-\alpha)t\}$$

For all $x, y, a \in X$, p ≥ 0, $\alpha\epsilon$ (0,2), $t > 0$ and $0 < s, q, r, q', r' = 1$ such that $2s = q + q' = r + r'$

(1.3) If one of $P(X), ABIL(X), STJU(X), Q(X)$ is a complete subspace of X then

1- P and $STJU$ have a coincidence point and
2- Q and $ABIL$ have a coincidence point.

Further if

(1.4) $AB = BA, AI = IA, AL = LA, BI = IB, BL = LB,\; IL = LI$
$QL = LQ, QI = IQ, QB = BQ,\; ST = TS, SJ = JS, SU = US,$
$TJ = JT,\; TU = UT, JU = UJ, PU = UP, PJ = JP, PT = TP$

(1.5) The pairs $\{P, STJU\}$ and $\{Q, ABIL\}$ are weakly compatible, then

1- $A, B, S, T, I, J, L, U, P \; and \; Q$ have a unique common fixed point in X.

Proof

By (1.1) since $P(X) \subset ABIL(X)$ for any point $x_0 \in X$ there exists a point x_1 in X such that $Px_0 = ABILx_1$. Since$Q(X) \subset STJU(X)$, for this point x_1 we can choose a point x_2 in X such that $Qx_1 = STJUx_2$ and so on. Inductively, we can define a sequence$\{y_{n\}}$ in X such that for $n = 0,1,2$…………..

$\boldsymbol{y_{2n}} = \mathrm{P}\boldsymbol{x_{2n}} = \mathrm{ABILx_{2n+1}}$ and $y_{2n+1} = Qx_{2n+1} = STJUx_{2n+2}$

We denote $d_n(t) = M(y_n, y_{n+1}, a, t)$ then by (1.2), for all t>0 and $\alpha = 1 - q$ with $q \in (0,1)$, wehave

$$[1 + pM(y_{2n}, y_{2n+1}, a, kt)] * M^{2s}\,(y_{2n+1}, y_{2n+2}, a, kt)] \geq p[M(y_{2n+2}, y_{2n+1}, a, kt) * M(y_{2n+1}, y_{2n}, a, kt) + M(y_{2n+1}, y_{2n+1}, a, kt) * M(y_{2n+2}, y_{2n}, a, kt)] + \min\{M^{2s}\,(y_{2n} + y_{2n+1}, a, t),\; M^q\,(y_{2n+2}, y_{2n+1}, a, t) .\, M^{q'}(y_{2n+1}, y_{2n}, a, t), M^r\,(y_{2n+1}, y_{2n+1}, a, (1-q)t). M^{r'}(y_{2n}, y_{2n+2}, a, (1-q)t\}$$

Now using (FM-4), on the lines of Sharma [223], we write

$$M(y_{2n+1}, y_{2n+2}, a, kt) \geq \min\{M^{2s}(y_{2n}, y_{2n+1}, a, t),\; M^q(y_{2n+2}, y_{2n+1}, a, t).\; M^{q'}(y_{2n+1}, y_{2n}, a, t), M^{r'}(x_{2n}, x_{2n+2}, x_{2n+1}, t/3q^n) * M^{r'}(x_{2n}, x_{2n+1}, a, t/3\,q^n) * M^{r'}(x_{2n+1}, x_{2n+2}, a, t/3\,q^n)\},$$

For every positive integer n in N, by noting that $M(x_{2n}, x_{2n+1}, x_{2n+2}, t/3q^n) \to 1 as\, n \to \infty$

Since the t-norm $*$ is continuous and $M(x, y, .)$ is continuous, we have $M(y_{2n+1}, y_{2n+2}, kt)$ $\geq \min\{M^{2s}(y_{2n}, y_{2n+1}, a, t), M^q(y_{2n+2}, y_{2n+1}, a, t). M^{q'}(y_{2n+1}, y_{2n}, a, t)\},$

So we have

$$d_{2n+1}^{2s}(kt) \geq \min\{d_{2n}^{2s}(t), d_{2n+1}^{q}(t), d_{2n}^{q}(t)\}$$

If $d_{2n+1} < d_{2n}$ then

$$d_{2n+1}^{2s}(kt) \geq d_{2n+1}^{2s}(t) \geq d_{2n+1}^{2s}(kt),$$

a contradiction, then $d_{2n+1}(t) \geq d_{2n}(t)$.

Similarly we can prove that $d_{2n+1}(t) \geq d_{2n+1}(t)$

Consequently $\{d_n\}$ is increasing sequence of non-negative real. Thus using

(1.2), we have

$d_{2n+1}^{2s}(kt) \geq d_{2n}^{2s}(t)$ that is $d_{2n+1}(kt) \geq d_{2n}(t)$.

Similarly, we have $d_{2n+2}^{2s}(kt) \geq d_{2n+1}^{2s}(t)$.

Thus for m=1,2,……..$M(\,y_{m+1}\,, y_{m+2}\,, a, kt\,) \geq M(\,y_m, y_{m+1}, a, t\,)$.

Hence by lemma 2, $\{y_n\}$ is a Cauchy sequence in X. now suppose $STJU(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $STJU(X)$ and has a limit in $STJU(X)$ call it z. let $\in STJU^{-1}(z)$. Then $STJUw = z$. we shall use the fact that subsequence $\{y_{2n}\}$ also converges to z. by (1.2), with $\propto = 1$ we have

By putting $x = w, y = x_{2n+1}$ in (1.2) with $\alpha = 1$ and taking limit as $n \to \infty$ we have

$$[1 + pM(STJUw, ABILx_{2n+1}, a, kt)] * M^{2s} (Pw, Qx_{2n+1}, a, kt)]$$
$$\geq p[M(Pw, STJUw, a, kt) * M(Qx_{2n+1}, ABILx_{2n+1}, a, kt) + M(Qx_{2n+1}, STJUw, a, kt) * M(Pw, ABILx_{2n}, a, kt)]$$
$$+ \min\{M^{2s} (ABILx_{2n+1}, STJUw, a, t),\ M^q (Pw, STJUw, a, t) . M^{q'} (Qx_{2n+1}, ABILx_{2n+1}, a, t), M^r (Qx_{2n+1}, SIJUw, a, t). M^{r'} (Pw, ABILx_{2n+1}, a, t)\},$$

$$[1 + pM(z, z, a, kt)] * M^{2s} (Pw, z, a, kt)]$$
$$\geq p[M(Pw, z, a, kt) * M(z, z, a, kt) + M(z, z, a, kt) * M(Pw, z, a, kt)]$$
$$+ \min\{M^{2s} (z, z, a, t),\ M^q (Pw, z, a, t) . M^{q'} (z, z, a, t), M^r (z, z, a, t). M^{r'} (Pw, , a, t)\},$$

$$M^{2s} (Pw, z, a, kt)] \geq M^{2s} (Pw, z, a, kt)$$

Therefore by lemma 3, we have $pw = z$. Since $STJUw = z$. Thus we have $pw = z = STJU$ that is w is coincidence point of P and $STJU$. This proves (i).

Since $P(X) \subset ABIL(X), pw = z$ implies that $z \in ABIL$. Let $v \in ABIL^{-1}z$ .Then $ABILv = z$.

By putting $x = x_{2n+2},\ y = v$ in (1.2), with $\alpha = 1$ and taking limit as $n \to \infty$ we have

$$M^{2s} (Qv, z, a, kt)] \geq M^{2s} (Qv, z, a, kt)$$

Therefore the lemma 3 we have $Qv = z$. Since $ABILv = z$, we have $Qv = z = ABILv$ that is v is coincidence point of Q and $ABIL$. This proves (ii).

The remaining two cases pertain essentially to the previous cases. Indeed if P(X) or Q(X) is complete then by (1.1), z P(X) ABIL(X) or z Q(X) STJU(X).

Thus (i) and (ii) are completely established.

Since the pair {P, STJU} is weakly compatible therefore P and $STJU$ commute at their coincidence point that is $P(STJUw) = (STJU)Pw$ or $Pz = STJUz$.

Since the pair $\{Q, ABIL\}$ is weakly compatible therefore Q and $ABIL$ commute at their coincidence point that is $Q(ABILv) = (ABIL)Qv$ or $Qz = ABILz$.

By putting $x = z, y = x_{2n+1}$ in (1.2) with $\alpha = 1$ and taking limit as $n \to \infty$ we have $M^{2s} (Pz, z, a, kt)] \geq M^{2s} (Pz, z, a, t)$.

Therefore by lemma 3, we have $Pz = z$. So $Pz = STJUz = z$. By putting $x = x_{2n+2}, y = z$ in (1.2), with $\alpha = 1$ and taking limit as $n \to \infty$ we have

$$M^{2s} (z, Qz, a, kt)] \geq M^{2s} (Qz, z, a, t)$$

Therefore by lemma 3, we have $Qz = z$. So $Qz = ABILz = z$. By putting $x = z, y = Lz$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$M^{2s} (z, Lz, a, kt)] \geq M^{2s} (Lz, z, a, t)$$

Therefore by lemma 3, we have $Lz = z$. So $ABILz = z$. By putting $x = z, y = Iz$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$M^{2s} (Iz, z, a, kt)] \geq M^{2s} (Iz, z, a, t)$$

Therefore by lemma 3, we have $Iz = z$. So $ABIz = z$ therefore $ABz = z$. Now to prove $Bz = z$ we put $x = z, y = Bz$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$M^{2s} (z, Bz, a, kt)] \geq M^{2s} (Bz, z, a, t) .$$

Therefore by lemma 3, we have $Bz = z$. Since $ABz = z$ therefore $Az = z$. To prove $Uz = z$ we put $x = Uz, y = z$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$M^{2s} (Uz, z, a, kt)] \geq M^{2s} (Uz, z, a, t) .$$

Therefore by lemma 3, we have $Uz = z$. Since $SIJUz = z$ therefore $SIJz = z$. To prove $Jz = z$ we put $x = Jz, y = z$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$M^{2s} (Jz, z, a, kt)] \geq M^{2s} (Jz, z, a, t) .$$

Therefore by lemma 3, we have $Jz = z$. Since $STJz = z$ therefore $STz = z$. To prove $Tz = z$ we put $x = Tz, y = z$ in (1.2), with $\alpha = 1$ and using (1.4), we have

$$M^{2s} (Tz, z, a, kt)] \geq M^{2s} (Tz, z, a, t). .$$

Therefore by lemma 3, we have $Tz = z$. Since $STz = z$ therefore $Sz = z$. By combinig the above results we have $Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z$. that is z is a common fixed point of $A, B, S, T, I, J, L, U, P$ and Q. The uniqueness of the common fixed point of $A,\ B,\ S,\ T,\ I,\ J,\ U,\ P$ and Q follows easily from (1.2). This completes the proof.

From theorem 1, with $p = 0$, we have the following result:

Corollary

Let $(X, M, *)$ be an fuzzy 2-metric space. Let $A, B, S, T, I, J, L, U, P$ and Q be mappings from X into itself satisfy condition (1.2) with $p = 0$. If conditions (1.1) and (1.3) are satisfied then conclusions (i) and (ii) of theorem 1 hold. Further if conditions (1.4) and (1.5) are satisfied then conclusion (iii) of theorem 1 hold.

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